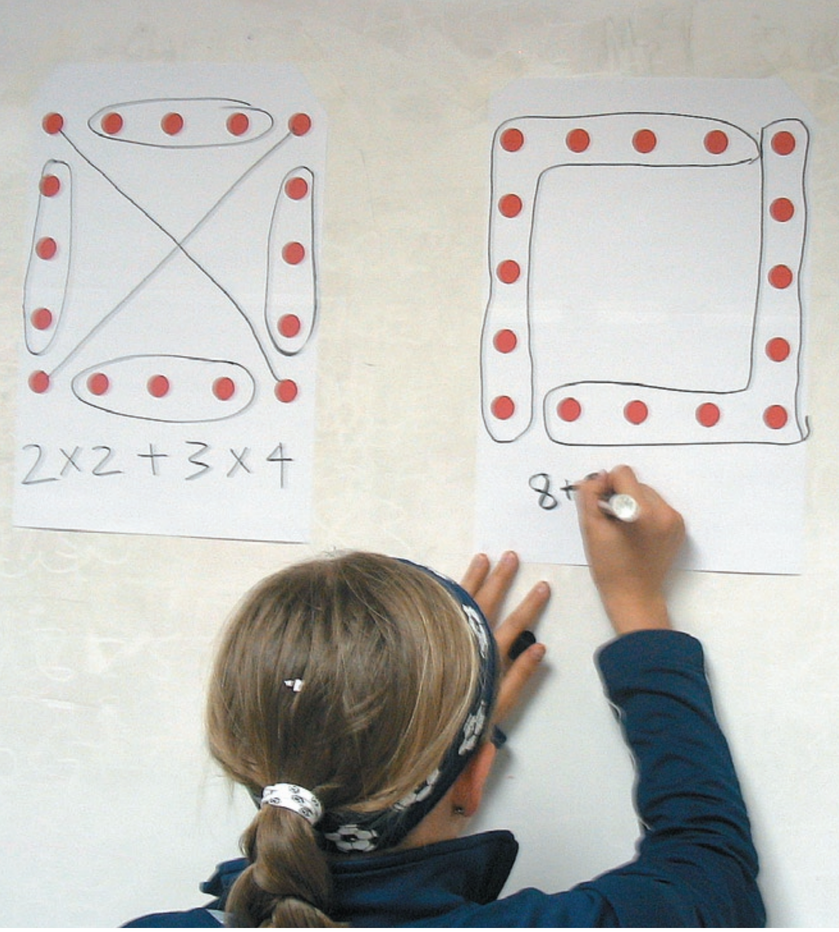

BLAKE E. PETERSON

Counting Dots and Measuring Area:

Rich Problems from Japan

IN FALL 2003, I HAD THE OPPORTUNITY TO conduct some research on the student-teaching process in Japan. During my seven weeks of research at the junior high school affiliated with Ehime University in Matsuyama, Japan, I observed mathematics lessons taught by student teachers as well as many more lessons taught by experienced teachers. The basis for most of these lessons was wonderfully rich mathematics problems. In these lessons, a problem was posed to the students, time was given for them to explore the problem, and then solutions were discussed. Similar problem-based lessons can be found in *The Teaching Gap* (Stigler and Hiebert 1999) and *The Open-Ended Approach: A New Proposal for Teaching Mathematics* (Becker and Shimada 1997).

Some assets of these problems were the connections students were able to make and the variety of representations they were able to employ in solving them. For example, connections were made between geometric, numeric, and algebraic patterns; between tabular, graphic, and symbolic representations; and between geometric behavior and algebraic functions. Over time, I began to realize that the richness of these problems had a great deal to do with the connections and representations that were such a prominent part of these Japanese lessons.

Many teachers in the United States are making efforts to incorporate the Process Standards from *Principles and Standards for School Mathematics* (NCTM 2000) into teaching and learning mathematics in their classrooms. The Problem Solving Standard for Grades 6–8 states, "Problem solving is central to inquiry and application and should be interwoven throughout the mathematics curriculum to provide a context for learning and applying mathematical ideas" (NCTM 2000, p. 256). The Connections Standard for Grades 6–8 states, "If curriculum and instruction focus on mathematics as a discipline of connected ideas, students learn to expect mathematical ideas to be related. Rich mathematical tasks prompt students to use and develop mathematical understandings and connections" (NCTM 2000, p. 275). If students are only presented with routine exercises that focus on a narrowly defined skill, connections are difficult to make. Solving broad, open-ended problems, however, allows students to see connections as part of the problem-solving process. Open-ended problem-solving situations also afford students the opportunity to use and move between various representations.

Rich problems, like the ones I observed in Japan, are excellent sources on which to build lessons that incorporate the Process Standards of Problem Solving, Connections, and Representation. In this article, I introduce two of my favorite problems that I saw

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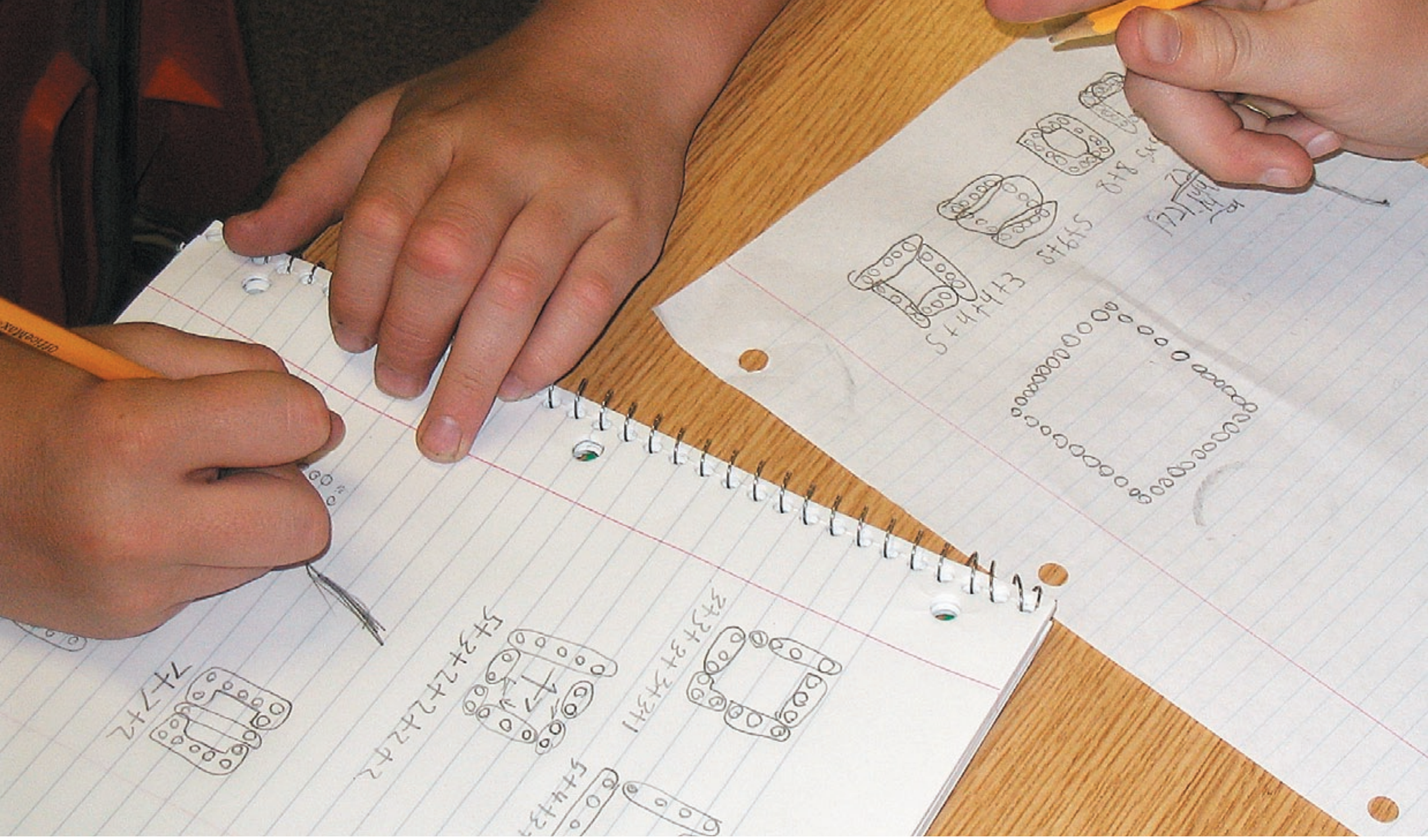
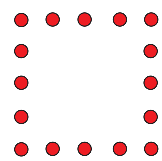


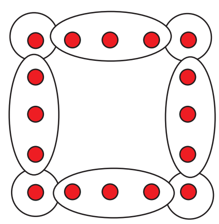
used by my colleagues in Japan, then describe how these problems play out in the classroom. Common student approaches to these problems will be presented, and where they might fit into the curriculum will be discussed.

The first problem examines the different ways that an arrangement of dots can be counted and the numeric expression associated with each counting method. The arrangement of dots is then expanded, and variables are introduced as a way to describe each counting method. The second problem, in which students are asked to use tables, graphs, and equations, naturally generates a piecewise linear equation from a geometric setting. The conversations about the different representations allow students to make many of the types of connections described above.



Problem 1: How many dots are in the diagram, and how is the total determined?

Fig. 1 The Dot problem



$$\Rightarrow \begin{cases} 4\times 3+4 \\ 4\times(5-2)+4 \end{cases} = 16$$

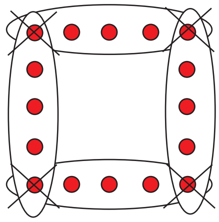
Fig. 2 Sides plus corners

Counting Dots and Writing Mathematical Expressions

THIS FIRST PROBLEM IS GEOMETRIC IN NATURE, where the geometry gives rise to equivalent mathematical sentences. As the geometric pattern is generalized, various algebraic representations emerge (see **fig. 1**). Many methods can be used to solve this problem. The discussion and comparison of these methods make this a rich problem. Once these methods have been discussed and organized, the students can be asked to generalize the situation. I will first discuss the methods that I have seen students use, then address the generalization.

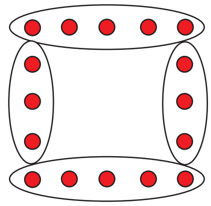
The initial response to this problem is 16 dots. When students were asked how they found that number, they either responded with an equation or how they visualized counting the dots. Each method outlined here is depicted by a visual image or picture. In many cases, multiple expressions or equations are associated with each method.

One method takes the interior dots on each side, 3, and multiplies by the number of sides, 4, then adds the dots in the corners, 4, to generate the expression



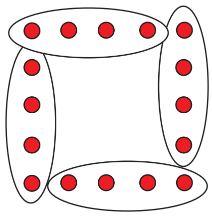
$$\Rightarrow \left\{ 4\times5-4=16 \right.$$

Fig. 3 Total sides less duplicate corners



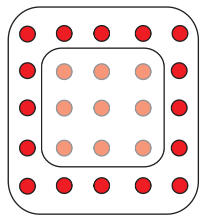
$$\Rightarrow \left\{ \begin{array}{l} 5+5+3+3 \\ 2\times5+2\times3 \\ 2\times5+2\times(5-2) \end{array} \right. =16$$

Fig. 5 Pairs of opposite sides



$$\Rightarrow \left\{ \begin{array}{l} 4\times4 \\ 4\times(5-1) \end{array} \right. =16$$

Fig. 4 Each corner assigned to one side



$$\Rightarrow \left\{ \begin{array}{l} 5\times5-3\times3 \\ 5^2-3^2 \\ 5^2-(5-2)^2 \end{array} \right. =16$$

Fig. 6 Total array less the interior

$4 \times 3 + 4 = 16$ (see **fig. 2**). Another way to write this expression is $4 \times (5 - 2) + 4 = 16$, since the 3 interior dots are determined by subtracting the two corners from the 5 dots on a side. Both of these expressions can be pictured in the same way, as shown in **figure 2**.

Another method is to multiply the number of dots on a side, 5, by the number of sides, then subtract the 4 corners that have been counted twice. This method generates the image and associated expression shown in **figure 3**.

So far, the corners seem to be the problem in the counting. Either the corners are not counted and have to be added at the end, or they are counted twice and one set has to be subtracted at the end. This third method handles the corners in the middle of the process instead of at the end. Simply assign each corner to one side. If this is done, 4 dots are counted on each side. Multiplying these 4 dots on each side by the number of sides yields the expression $4 \times 4 = 16$ (see **fig. 4**). Because the 4 dots on each side come from the 5 total dots on a side minus 1 corner, this expression could also be written as $4 \times (5 - 1) = 16$. The geometric origins of both of these expressions are drawn in the same way, as shown in **figure 4**.

This next method also handles the corners nicely by viewing pairs of opposite sides, as shown in **figure 5**. Two sides include two corners each, and the other two sides include no corners. Using this method of viewing the dots yields the expression $5 + 5 + 3 + 3 = 16$. This expression can be written in two other ways (see **fig. 5**).

This next method uses a very different perspective on the figure by counting the 5×5 array of 25 dots, then subtracting the 3×3 array of 9 dots that fills the middle (see **fig. 6**). This numerical expression would be $5 \times 5 - 3 \times 3 = 16$, or $5^2 - 3^2 = 16$. Once again, the 3 is determined by subtracting 1 dot from each end of a row of 5 dots so this expression could be written as $5^2 - (5 - 2)^2 = 16$. The geometric image associated with this method and the corresponding numerical expressions are shown in **figure 6**.

The natural progression of counting the dots in each of the situations involves visualizing the groupings of dots, like those shown for each of the methods so far, then generating a numerical expression that corresponds to that visual image. When students share their solutions, however, they usually provide the numerical expression first, then have to be asked what is the corresponding visual image. In one case, when the Japanese students suggested the expression $2^4 = 16$, the teacher had never seen it before and doubted that a geometric image corresponded to it. When I saw the lesson, I felt the same way and thought that this expression had likely been generated from one of the other numerical expressions rather than from a geometric image. The teacher and I were both proved wrong when one seventh-grade student raised her hand and claimed that there was a geometric interpretation of it. She came to the board and sketched **figure 7d**. **Figures 7a–c** show the order in which the student drew the circles. She also described how the circled dots corresponded to the numerical expression. **Figure 7a** has 2 dots circled. **Figure 7b** has two groups of 2 dots, or 2^2, circled. **Figure 7c** shows two groups of 2^2 dots, or 2^3, circled. Finally, **figure 7d** shows two groups of 2^3 dots, or $2^4 = 16$, circled.

Once the various methods of counting the dots have been shared, a discussion of the similarities and differences among the methods and the numerical expressions as well as the origins of the numbers in each expression occurs. For example, a "4" in an ex-

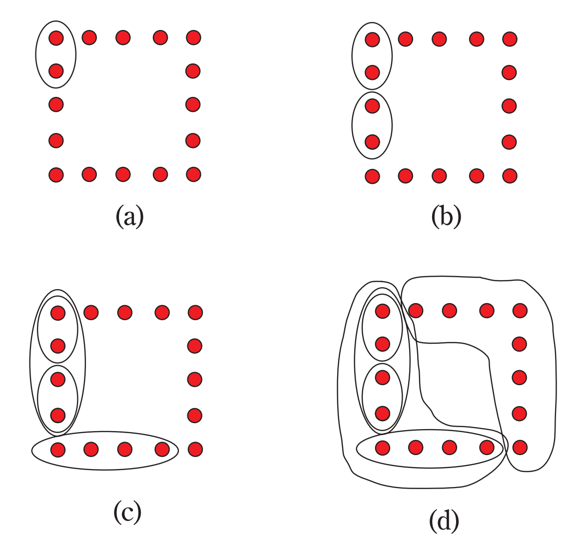
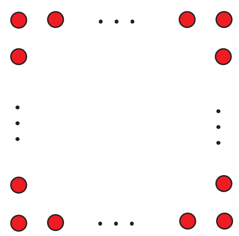


Fig. 7 Geometric representations of $2^4 = 16$



Follow-up problem: What if you had 25 dots or 100 dots or n dots on each side of the square? (See the diagram above.) How many total dots would there be?

Fig. 8 A generalization of the pattern

pression could be the number of sides or the number of corners or 1 less than the number of dots on a side. This conversation helps students better connect the geometric situation with the numeric expressions and makes it much easier to write general expressions. Another class discussion can focus on the fact that all the mathematics sentences are equivalent because they are all equal to 16. A follow-up problem is then stated to allow the students to generalize the situation and write algebraic representations of the various methods of counting the dots (see **fig. 8**).

For the students, it is best to consider cases of a large number of dots on a side (25 or 100) before considering the general case. The students should be challenged to use several of the methods discussed to find the number of dots in a similar square figure with a large number of dots on each side. For our purposes, we will move directly to n dots on a side.

On one hand, since the number of sides, 4, and the number of corners, 4, will be the same regardless of the number of dots on a side, the values that represent the number of sides or number of corners will stay fixed at 4. On the other hand, any value that was determined based on the number of dots on the side will have to reflect the value of n. By understanding the role that each number plays in an expression, it becomes easier to write a generalization using variables. Thus, the expressions that represent the total number of dots on a square figure with n dots on a side would be written as follows:

$$\left.\begin{array}{l} 4\times 3+4 \\ 4\times(5-2)+4 \end{array}\right\} \Rightarrow 4(n-2)+4$$

$$4\times 5-4 \Rightarrow 4n-4$$

$$\left.\begin{array}{l} 4\times 4 \\ 4\times(5-1) \end{array}\right\} \Rightarrow 4(n-1)$$

$$\left.\begin{array}{l} 5+5+3+3 \\ 2\times 5+2\times 3 \\ 2\times 5+2\times(5-2) \end{array}\right\} \Rightarrow 2n+2(n-2)$$

$$\left.\begin{array}{l} 5\times 5-3\times 3 \\ 5^2-3^2 \\ 5^2-(5-2)^2 \end{array}\right\} \Rightarrow n^2-(n-2)^2$$

Each algebraic expression above corresponds to a method of computation described earlier. The method shown in **figure 7**, however, is more complicated, so the algebraic generalization is not shown.

Finally, since all the expressions on the left are equal to 16, all the expressions on the right should be equivalent. My Japanese colleague used the equivalence of these expressions to motivate students to use the properties of numbers and equations to manipulate these expressions and see that they are equal to one another. Thus, this problem leads to a rich context-based exploration of various algebraic procedures.

Measuring Area and Graphing Piecewise Linear Functions

FREQUENTLY, WHEN PIECEWISE LINEAR FUNCTIONS are introduced to algebra students, they are presented in symbolic form, and students are asked to generate graphs of the functions. These functions are usually contrived, and students wonder if such a situation would ever arise in real life. The next problem about a geometric situation gives rise to a piecewise linear function. It not only provides a nice connection between geometric and algebraic representations but gives students a chance to see a

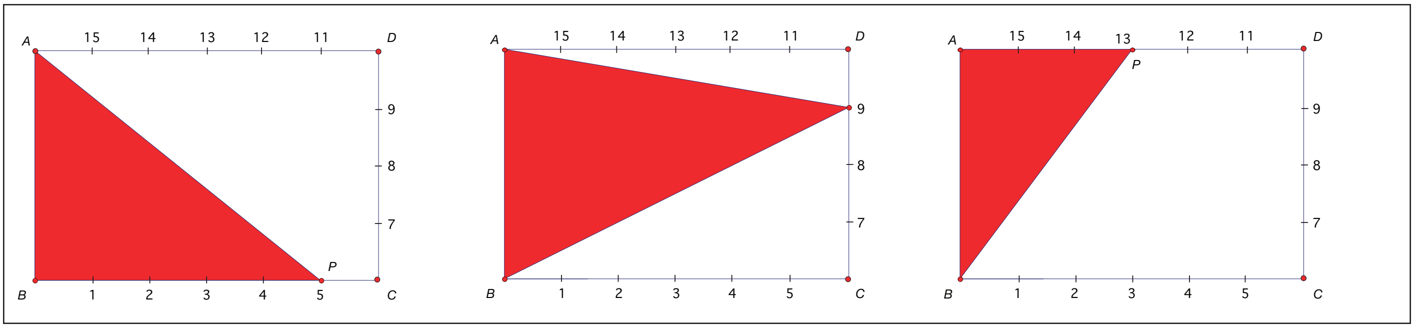


Fig. 9 Triangles at 5, 9, and 13 seconds

x (sec)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
y (cm²)	0	2	4	6	8	10	12	12	12	12	12	10	8	6	4	2	0

Fig. 10 Relationship between time and area

physical model of a piecewise linear function.

This problem can be posed in a variety of different ways depending on the level of the students. In some cases, more detail and support are provided; in others, the problem is posed in a more open way. Rather than prescribe the detailed way in which this problem could be structured, I pose it in a more general format as seen in the worksheet and leave it to the teacher to structure posing the problem in a way that meets the needs of their students. (See the worksheet **Measuring Area and Graphing Piecewise Linear Functions** at the end of this article.)

The solution to number 1 in this problem is found by sketching some of the triangle configurations for different time frames. Having the student sketch the triangles at different time intervals is a great beginning point. Examples of the triangles at 5 seconds, 9 seconds, and 13 seconds are shown in **figure 9**. The corresponding area for each second in the interval from 0 to 16 is then determined, and the data are organized in a table (see **fig. 10**). These data reveal that the area is increasing for the first 6 seconds, is constant for 4 seconds, and decreases for the last 6 seconds.

As students share their final solutions, it is important to discuss how the students know that this equation is linear just by looking at the table of values. Recognizing that the constant differences in the three intervals are indicative of the linear slope is a valuable connection.

By plotting points and using the data from **figure 9** to identify the constant differences in each time interval as linear behavior, a graph can easily be constructed to answer number 2 on the worksheet (see **fig. 11**).

Because the graph increases initially and later decreases, some students may think that it is a parabola. Reconciling that notion with the numerical pattern in the table, which appears to be linear over each of three intervals, can be challenging.

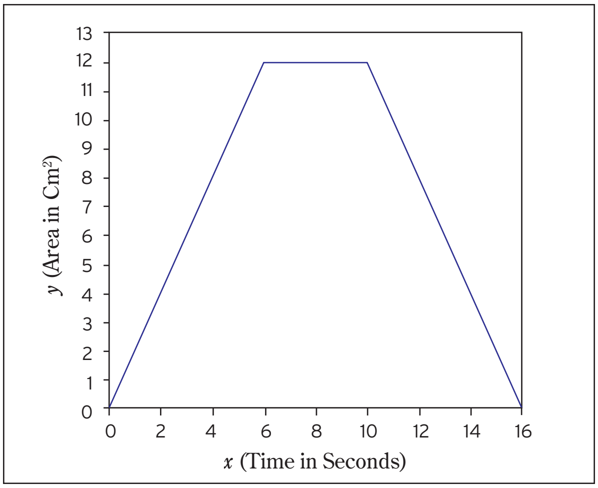


Fig. 11 A graph of the time and area relationship

For question 3, an equation needs to be generated based on the data table and the graph. Although students can see from both that the relationship behaves differently in different time intervals, they may initially struggle writing an equation. Their previous experience may indicate that a graph must be described with only one equation. Once they recognize that it requires different equations for each of the three intervals, writing the equation for the graph becomes much easier.

These intervals are $0 \le x \le 6$; $6 < x \le 10$; and $10 < x \le 16$. In each interval, the slope can be easily determined by noting that for consecutive seconds the difference between the areas is constant. The constant difference is the slope. For example, from 0 to 6 seconds, the area increases 2 square centimeters for each second. Thus, the difference in consecutive y values in the first interval is 2, implying that the slope is 2 on that interval. In a similar way, the slope can be determined for each of the other two intervals. As mentioned earlier, connecting constant dif-

ferences in tabular values with the slope of an equation is useful in understanding linear behavior.

These equations result:

$$y = \begin{cases} 2x & 0 \le x \le 6 \\ 12 & 6 < x \le 10 \\ -2x + 32 & 10 < x \le 16 \end{cases}$$

One great asset of this problem is the fact that it generates a piecewise linear graph in a natural way. Requiring the students to use a table, a graph, and an equation creates an opportunity to see the connections among them. Having students share and justify their solutions with other class members gives them further opportunities to communicate in ways described in the Process Standard of Communication.

Conclusion

WHEN MAKING CONNECTIONS, "PROBLEM SELECTION is especially important because students are unlikely to learn to make connections unless they are working on problems or situations that have the potential for suggesting such linkages" (NCTM 2000, p. 359). The first problem allows students to make natural connections between the geometric arrangement of dots and the numeric expressions related to each counting method. The generalization to *n* dots on a side follows from the connections made.

The second problem centers around the representations of tables, graphs, and equations in which the students create a table, then a graph, and finally an equation. Since this is not the common order of generating a table from an equation, solving this problem should help the students move more flexibly among the various representations. The importance of learning about these representations is described in *Principles and Standards*:

> The study of linear functions, with the associated patterns and relationships, is another major focus in the middle grades. By considering problems in a variety of contexts, students should become familiar with a range of representations for linear relationships, including tables, graphs, and equations. Students need to learn to use these representations flexibly and appropriately. (NCTM 2000, p. 282)

Using problems like those discussed here provides a context that allows students the chance to make multiple connections without being told precisely what to look for. The use of geometric, numeric, tabular, graphic, and symbolic representations in these problems gives students the opportunity to see patterns that they might not normally see if they were studying only one or two representations independent of one another. These engaging items provide excellent problem-solving experiences, but what is more important is that they "give students the opportunity to learn important content through their explorations of the problems" (NCTM 2000, p. 341). These problems engaged students in building rich, connected ideas in multiple representations. It helped them build ideas that in the United States are often stated and memorized.

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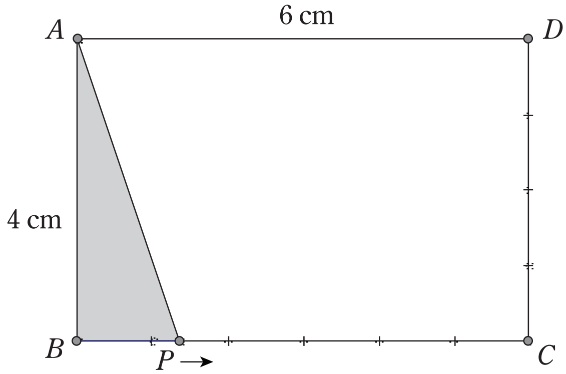
Measuring Area and Graphing Piecewise Linear Functions

NAME ______________________________

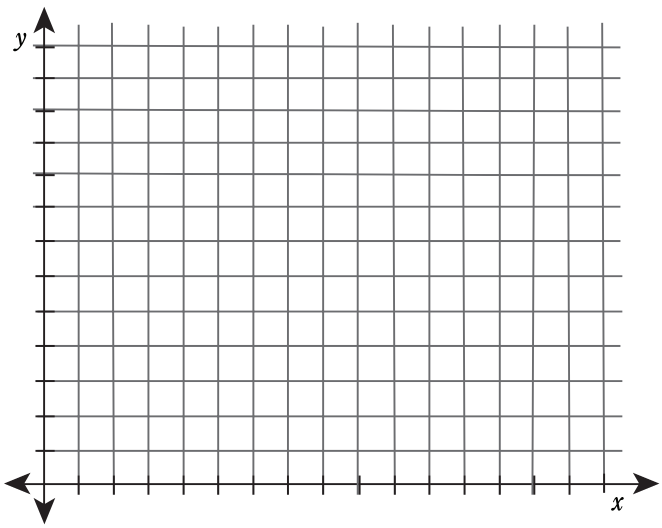
Consider rectangle *ABCD* with side lengths of 6 cm and 4 cm as shown. Point *P* will move along $\overline{BC}$ from point *B* to point *C*, along $\overline{CD}$ to point *D*, and finally along $\overline{DA}$ to point *A* at a constant rate of 1 cm per second. Let *y* be the area of $\triangle ABP$, and *x* be the seconds elapsed.



1. Complete the table showing the relationship between *x* and *y*.

x (sec)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
y (cm^2)																	

2. Construct a graph of *x* and *y*.



3. Write an equation to represent this relationship.

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